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Simulation of Viscous Fingering due to Saffman-Taylor Instability in Hele-Shaw Cell

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ABSTRACT

We simulated the growth of the viscous fingers in a Hele-Shaw cell, a Laplacian growth, by a numerical solution. We solved a Laplace equation numerically with boundary conditions that indicated a pressure jumping due to the surface tension in the interface of the two fluids. By using Darcy's law, we gained the time evolution of the interface and then visualized it in MATLAB. Subsequently, we examined the effects of several parameters in forming the fingers in rectangular cells and compared the results with the theoretical predictions which had a desirable agreement with our experimental findings. Our findings indicated that wave-lengths scale well with the control parameter in all conditions. Furthermore, we estimated the time evolution of the interface for Newtonian and non-Newtonian fluids in a circular cell, for Newtonian fluid, in agreement with experimental finding and theoretical prediction for dominant pattern was tip splitting. For non-Newtonian fluids from Shear-Thinning kind, we used two generalized Darcy's equation, we found that in both cases the tip dose not split but it will be sharped, finally we found that the two different model suggested for generalized Darcy's low (Bonn's model and Kondic's model) are in good agreement with each other and also with the experimental findings.

Keywords: Saffman-Taylor instability, Viscous Fingering, Yield Stress, Shear-Thinning, Darcy's Equation.

1. INTRODUCTION

Pushing a fluid with less viscosity into a fluid with more viscosity in a Hele-Shaw cell which results in Saffman-Taylor instability has recently attracted many scholars [1-3] because this instability is the offspring of the viscous fingering formation and refers to the appearance of finger-like interfacial patterns [2]. In this instability, [4] the growth of fingers is a sort of Laplacian growth. This interesting phenomenon is regarded as a representative of interfacial pattern formation and has been studied numerously, from various perspectives, since it also frequently occurs in nature and industrial applications, such as sugar refining, carbon sequestration, enhanced oil recovery [5], oil well cementing [9], printing devices [8], chromatographic separations [7], coating, adhesives, and growth of bacterial colonies [6]. Viscous fingering in a traditional Hele-Shaw cell [11-10], made of two parallel flat plates with a small gap, has received much attention as a suitable framework to analyze interfacial instabilities in narrow confined passages, e.g., in porous media [11]. The Saffman-Taylor instability problem for non-Newtonian fluids is not very well defined [12] due to complex rheological behaviors exhibited by these fluids. The effects of several key non-Newtonian properties have been investigated, such as yield stress [13, 1, 15, 16], shear-thinning [2, 9], shear-thickening [9] and elastic behaviors [12, 17]. New, diverse classes of problems have been discovered [17, 18], e.g., snowflake-like patterns [11] and branched, fractal, or fracture-like

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structures [19]. In particular, it has been found that shear-thinning effects induce dendritic patterns (with side branching) or crack-like patterns (with angular branches and sharp tips) [23, 3, 20]. Shear-thickening features may widen or narrow the finger width [21]. Viscoelastic properties have been found to strikingly modify Newtonian morphological patterns. For theses fluids, Lindner et al. [13] have discovered the existence of a yield stress regime (with ramified structures) and a viscous regime (with a single finger) at small and moderate velocities. Maleki et al. [1] have also observed a side-branching regime at larger velocities. Numerical simulations of Ebrahimi et al. [14] confirmed some of the observed behaviors for these fluids. Finally, a fluid's thixotropic drastically affects the finger shape, leading to chaotic behaviors at longer times [25].

Many experimental and theoretical works are performed in this regard and also are performing [1-23]. For example, Kondic performed several simulations in this case [22-23]. In a laboratory viscous fingering experiment, measurements are performed in a linear Hele-Shaw cell consisting of two glass plates separated by a thin spacer. The cell is filled with the viscous fluid, after which the less viscous fluid is forced (for instance, under pressure) into the channel. Usually, one quantifies the width of the finger (relative to the channel width) as a function of the finger velocity. Another example in [1] was observed different regimes that lead to different morphologies of the fingering patterns, in both rectangular and circular Hele–Shaw cells, which these results were in rather good agreement with the theoretical predictions. Another one experimentally works in the Saffman-Taylor instability of air invasion into a non-Newtonian fluid was studied by [22].

2. THOERETICAL FOUNDATION

Darcy's law obtains from replacing boundary conditions in the Hele-Shaw flow in the Navier-Stokes equation and exact analytical solutions for this equation are low. Laplace's equation for quasi-steady normal diffusion in the cylindrical and Cartesian coordinates, are respectively:

$$(1/\rho)(\partial/\partial\rho)(\rho\partial\phi/\partial\rho) + (1/\rho^2)(\partial^2\phi/\partial^2\varphi) = 0,\ \partial^2 p/\partial y^2 + \partial^2 p/\partial x^2 = 0 \quad (1)$$

In the quasi-steady approximation, we are solving $\partial\phi/\partial t = D\nabla 2\phi$ in a regime in which $\partial\varphi/\partial t$ can be neglected. The boundary $\partial\Omega z$ (t) is assumed to be an equi-potential, or constant-field, surface satisfying the boundary condition $\varphi = 0$. Alternatively, we can write (1) as:

$$\partial\phi/\partial t = -\nabla . F \quad (2)$$

Where the flux F is defined as

$$F = -D\nabla\varphi. \quad (3)$$

The phenomenon which satisfies this theory is known as Laplacian Growth Phenomena and a physical example of this process is viscous fingering in a Hele-Shaw cell [7]. Darcy's law in a porous medium or Hele-Shaw cell states that the velocity is given by $u = (-b2 / 12\eta)\nabla P$, where the velocity satisfies the continuity equation and the pressure satisfies Laplace's equation:

$$\nabla \cdot u = 0, \nabla 2P = 0. \quad (4)$$

That η is viscosity of fluid and b is spacing between plates of cell.

If we assume that we have a bubble of less viscous fluid at an essentially constant pressure $P = P0$ pumped into a more viscous fluid moving in a field u proportional to ∇P, then the velocity of the boundary is given by

$$v = u = \mu \nabla P. \tag{5}$$

Saffman and Taylor (1958) studied a quasi-two-dimensional system in which a fluid is trapped between two flat plates, known as Hele-Shaw cell. A second fluid is injected through a hole in the top of the system, which leads to a pressure gradient. The pressure gradient induces a velocity field; the height-averaged velocity is also given by (5), where μ is the viscosity of the fluid.

3. METHODS AND RESULT

3.1 Rectangular Cell

First Darcy's equation in the rectangular cell that in fact leads to the Laplace's equation for pressure in the Cartesian coordinates, i.e. $\partial 2p/\partial y2 + \partial 2p/\partial x2 = 0$ solved numerically with computer then with exercising the small perturbation ξeiky in the bound, we simulated the motion of the bound with numerical solution in the MATLAB. Then the following considerations were regarded:

1- The time evolution of fluid

We obtained the time evolution of Newtonian fluid and simulated fingering patterns that is shown in Figure 1. Underneath, the time evolution of several fluids is shown that every curve in the profiles shows bound of fluid in a moment.

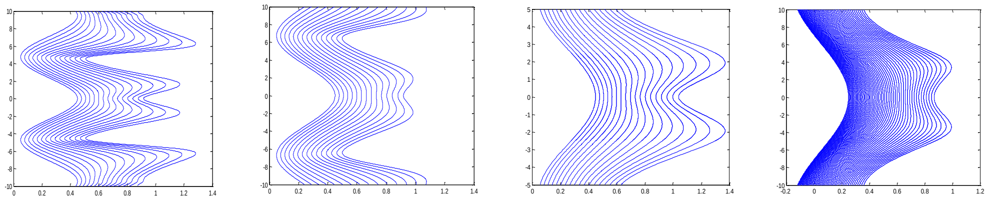

Figure 1. The time evolution of fingers.

2- Comparison with the theory by considering the effect of parameters in the Newtonian fluid

a. Effect of viscosity (η)

Several fluids that have only different viscosity have been considered. According to

$$\lambda = \pi b \sqrt{\frac{\sigma}{U(\eta - \eta')}} \tag{6}$$

by increasing viscosity, wave-length should decrease. Figure2-b shows a fluid that its viscosity is fifteen-fold of fluid shown in Figure 2-a. Both program outputs (Figures 2-a & 2-b) and graph 2-c uphold this subject.

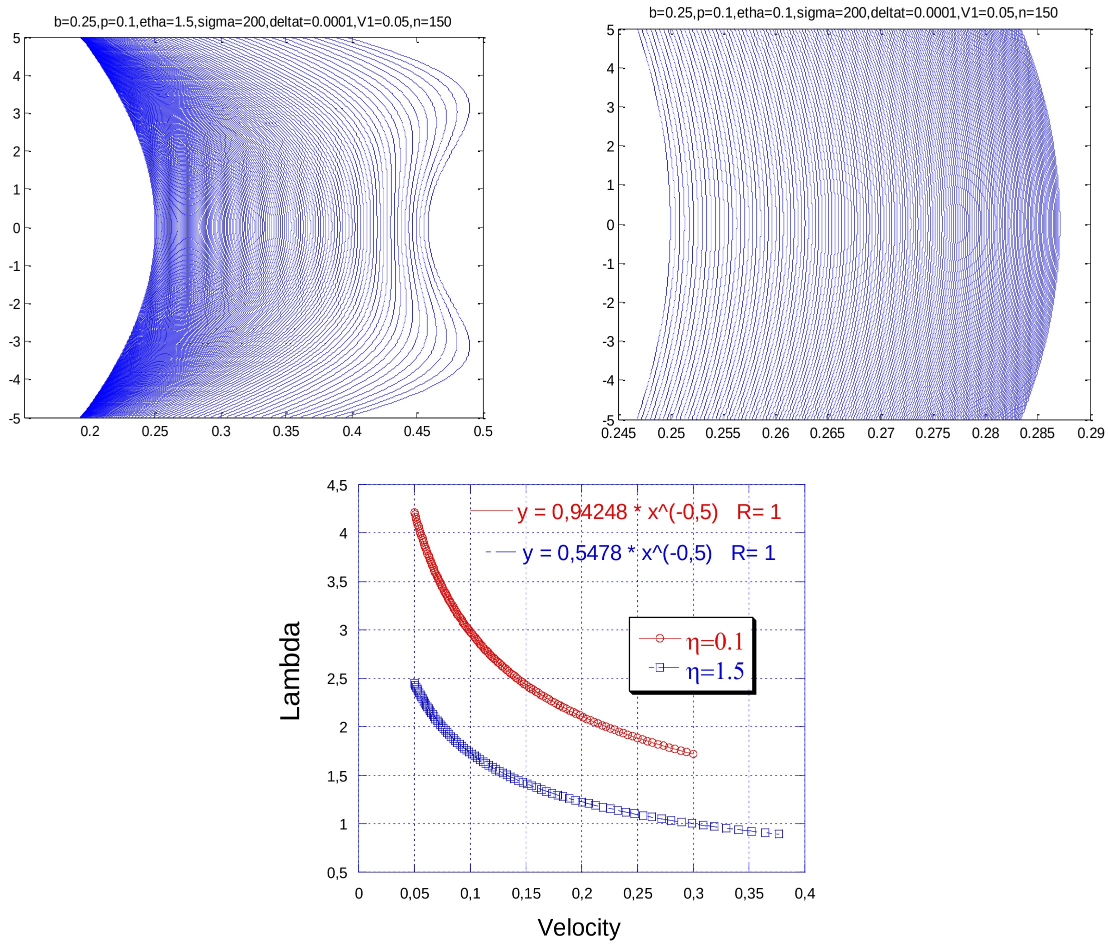

Figure 2. Output of program for (a) fluid with less viscosity and (b) fluid with more viscosity, (c) comparison of wave-lengths graph.

b. The effect of spacing between (b)

Several fluids that have only different spacing between the plates of Hele-Shaw cell have been considered. According to (6) by increasing b, wave-length should increase, Figure 3-a shows conditions that b for it is only two & half of a fold of Figure 3-b. program outputs (Figures 3-a & 3-b) and graph 3-c uphold this subject.

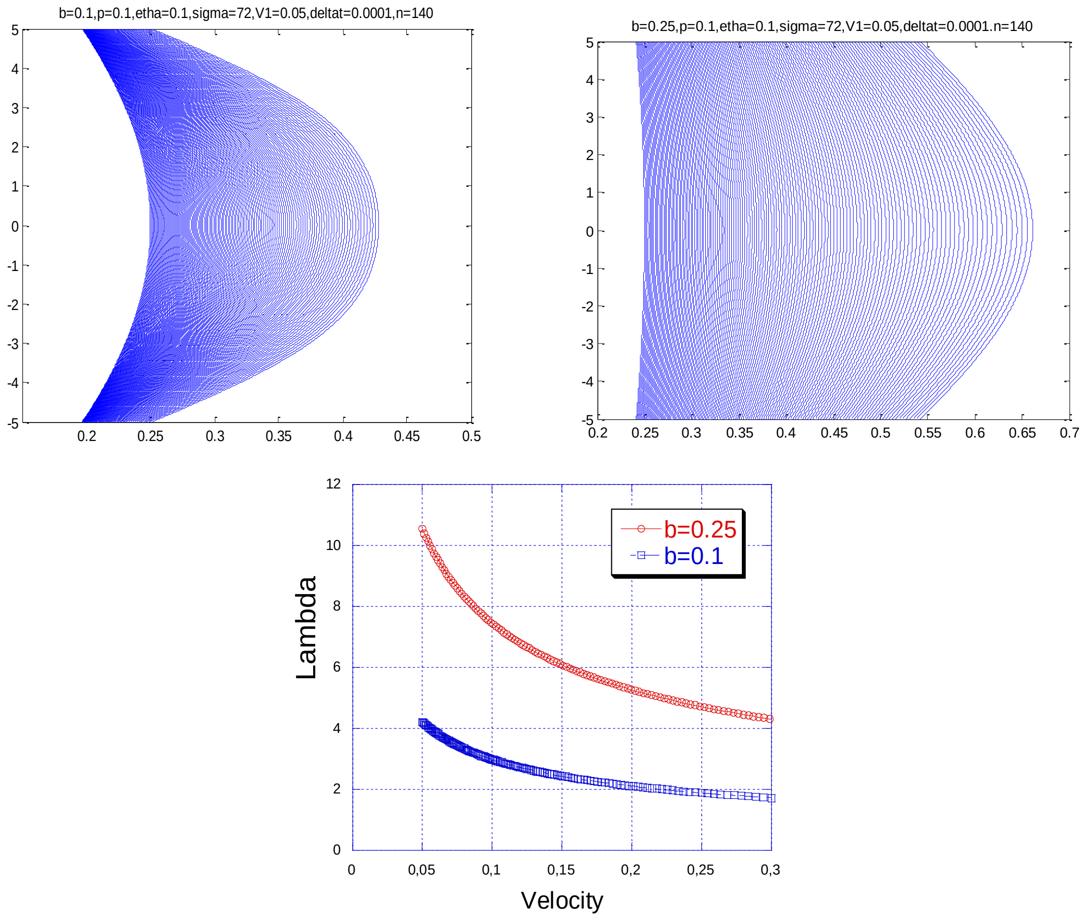

Figure 3. Output of program for (a) b=0.25 and (b) b=0.1, (c) comparison of wave-lengths graph.

c. The effect of pressure gradient on the fluid (p)

In this case, with all conditions are the same, only the pressure gradient on the fluids is different. The pressure gradient in the case shown in Figure 4-b, is for times of the pressure gradient for fluid in Figure 4-a. Figures 3-a, b & c show that the results of the program are in agreement with prediction of Saffman-Taylor theory. The higher pressure gradient accelerates the creation of perturbation and the formation of fingers.

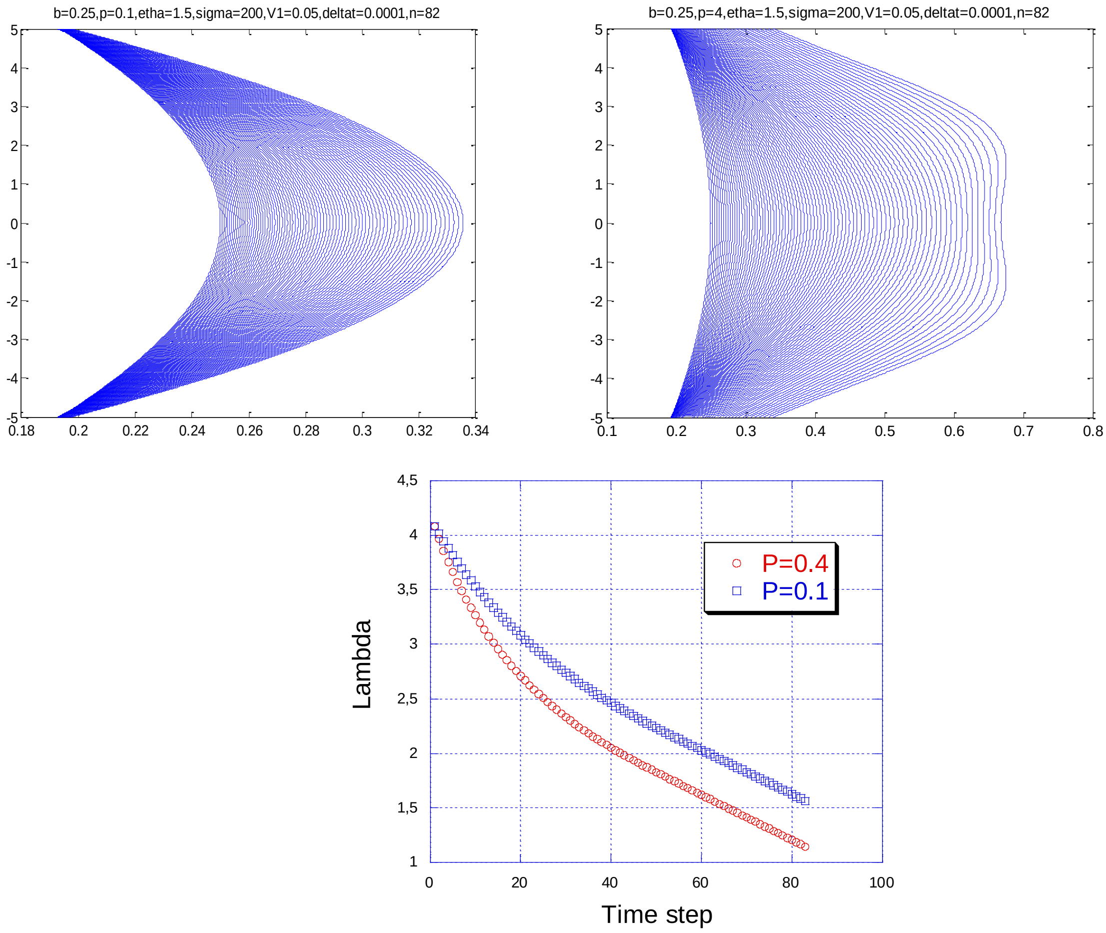

Figure 4. Exercised pressure is in a) 0.1 and b) 0.4. Graph c) shows the fluid in more pressure perturbs rather and forms finger.

d. Comparison with the theory by considering control parameter(1/B)

According to classical Saffman-Taylor instability, wave-length scales with control parameter defines as

$$1/B = (\sigma/12\eta U)\,(w/b) \tag{7}$$

W is width of cell. On the other hand graph wave-length of fingers independent of all parameters posed on the unit age curve control parameter. We performed this exam on our results, as can be seen in Figure 5 there is exactly a unique curve for wavelength in different conditions, which means the control parameter correctly and completely scales the wavelength, in agreement with the prediction of theory.

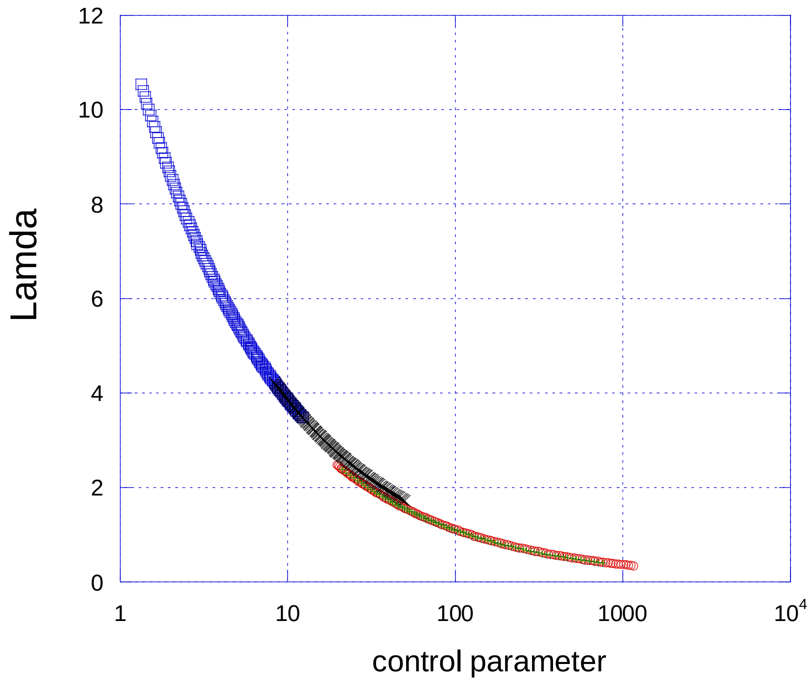

Figure 5. Wave-length graph unit age control parameter.

3.2 Circular Cell

3.2.1 Newtonian Fluid

Here again Darcy's equation in the circular cell that in fact leads to the Laplace's equation for pressure in the cylindrical coordinates, i.e. $(1/\rho)(\partial/\partial\rho)(\rho\partial\phi/\partial\rho)+(1/\rho^2)(\partial^2\phi/\partial^2\varphi) = 0$ is solved numerically with computer then with exercising the small perturbation $\xi\cos(4\theta)$ in the bound, we simulated the motion of the bound with numerical solution in the MATLAB and obtained Figure 6 for this fluid.

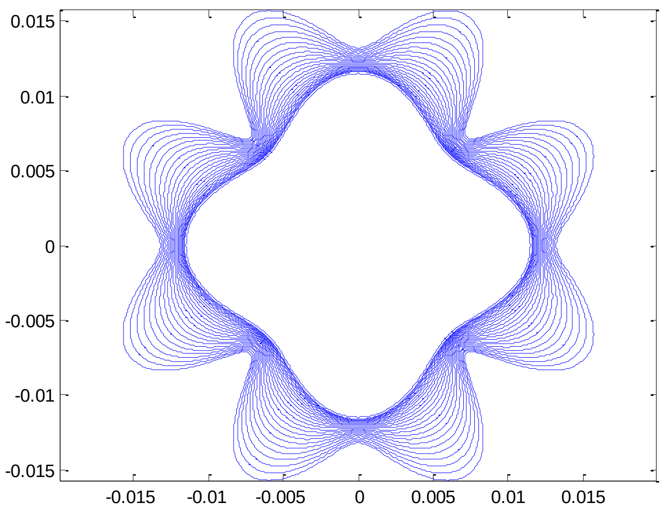

Figure 6. Time evolution of Newtonian fluid in the circular cell.

3.2.2 Non-Newtonian Fluid

a. Bonn's model

For the non-Newtonian fluid that its viscosity along flow varies, first we should adopt a model for the way of how viscosity along flow varies. Bonn and et al [9] presented a model that viscosity in it is variable and a function of shear rate is as:

$$v = \frac{-b^2}{12\mu(\dot{\gamma})} \nabla p \tag{7}$$

that $\dot{\gamma}$ is shear rate and pertains to velocity of fluid in the Hele-Shaw cell and b with equation $\dot{\gamma} = 3v/b$. They named the equation (7) the generalized Darcy's law [9]. We by using this model obtained Figure 7 for time evolution of a shear-thinning fluid that sharpening in the tip of finger well shows for this fluids likewise occurs in the laboratory.

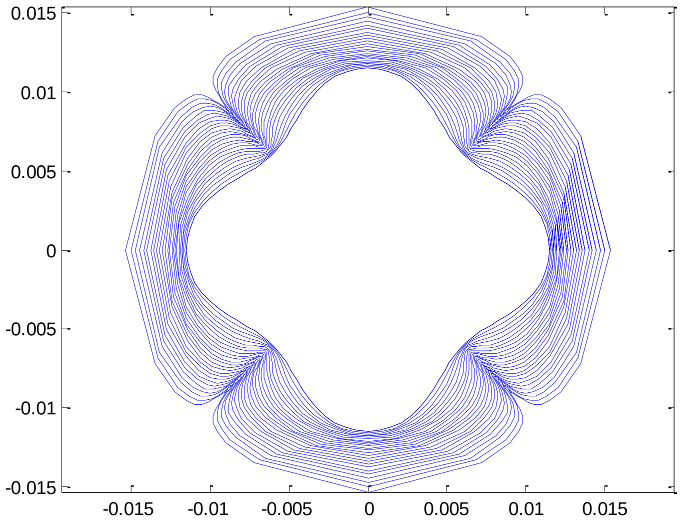

Figure 7. Time evolution of a non-Newtonian fluid in the circular cell based on Bonn's model.

b. Kondic's model

In this model presented by Kondic and et al [6] for non-Newtonian fluid, viscosity is supposed to be a function of pressure gradient so that Darcy's equation becomes

$$v = \frac{-b2}{12\mu\left(\left|\nabla p\right|^2\right)} \nabla p \tag{8}$$

We obtain Figure 8 for time evolution of a shear-thinning based on this model that is in good agreement with both Bonn's model and our experimental results.

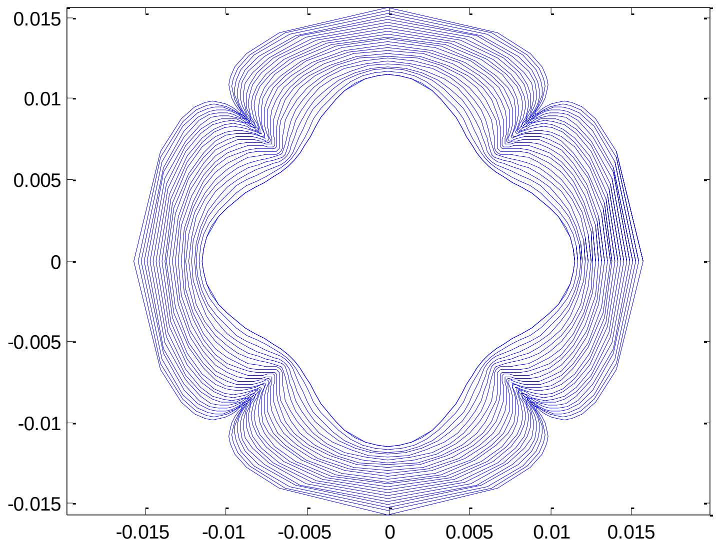

Figure 8. Time evolution of a non-Newtonian fluid in the circular cell based on Kondic's model.

4. CONCLUSION

We simulated the growth of viscous fingering due to Saffman-Taylor instability and could obtain results for Newtonian fluid in the rectangular cell that is in complete agreement with both experimental results on one hand and the prediction of Saffman-Taylor instability on the other. The parameters in this phenomenon have simulated in the computer affect in the same way that the theoretical foundations predict and according to what is observable in the laboratory. We also obtained time evolution of Newtonian and non-Newtonian fluid. For Newtonian fluid, the dominant pattern was tip splitting, in agreement with experimental results. For shear-thinning Fluids, we checked both generalized Darcy's equation, suggested by Bonn [24] and Kondic [22-23]. We found that in both cases, the tip dose not split but it will be sharpened, which are in agreement with experimental results reported for shear thinning fluids, at the same time, we found that these two different models(Bonn's and Kondic's models) suggested for generalized Darcy's low are in good agreement with each other as with the experimental findings. Finally we used a simple numerical method based on solving the Laplacian growth to show and analyze the viscous fingering process due to Saffman-Taylor instability. In this method the results are in good agreement with experiment, and states that the both Bonn's and Kondic"s model are the same.

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